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Twistor spaces of hyperkähler manifolds with S^1-actions

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Abstract

We shall describe the twistor space of a hyperkähler $4n$-manifold with an isometric S^1-action which is holomorphic for one of the complex structures, scales the corresponding holomorphic symplectic form and whose fixed point set has complex dimension n.

We deduce that any hyperkähler metric on the cotangent bundle of a real-analytic Kähler manifold which is compatible with the canonical holomorphic symplectic structure, extends the given Kähler metric and for which the S^1-action by scalar multiplication in the fibres is isometric is unique in a neighbourhood of the zero section. These metrics have been constructed independently by the author and Kaledin.

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1. Introduction

Hyperkähler manifolds are Riemannian manifolds which possess three orthogonal automorphisms I, J and K of the tangent bundle which are parallel with respect to the Levi-Civita connection and satisfy the relations of the quaternions

$$I^2 = J^2 = K^2 = IJK = -1.$$

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It is well known that hyperkähler manifolds only admit essentially two types of isometric S^1-actions: either the action is triholomorphic or it preserves one of the complex structures and rotates the other two. In this paper we investigate the twistor spaces for the latter type of circle action in the case when the fixed point set has dimension equal to half of the dimension of the hyperkähler manifold.

Examples of such hyperkähler manifolds are the hyperkähler metrics on a neighbourhood of the zero section of the cotangent bundle of any real-analytic Kähler manifold constructed in [3] and independently in [7]. The S^1-action by scalar multiplication in the fibres is isometric and holomorphic for the complex structure induced from the base. As the corresponding holomorphic symplectic form coincides with the canonical symplectic form of the cotangent bundle, this action scales the the holomorphic symplectic form.

Let M be a hyperkähler manifold with isometric S^1-action which is holomorphic with respect to the complex structure I and scales the corresponding holomorphic symplectic form. Let X be the fixed point set of the action and assume that $\dim X = \frac{1}{2}\dim M$.

Note that X is an I-complex submanifold of M and in fact Kähler with respect to the restriction of the hyperkähler metric on M to X. The Kähler metric is real-analytic since the hyperkähler metric is (as it is Einstein).

Any real-analytic Kähler manifold X possesses a hyperkähler structure in a neighbourhood of the zero section of the cotangent bundle T^*X, see [3] and independently [7], and we shall show that locally near X the hyperkähler structure on M looks like the hyperkähler structure on T^*X. More precisely,

Theorem A. *Let M be a hyperkähler manifold with isometric S^1-action which is holomorphic with respect to the complex structure I and scales the corresponding holomorphic symplectic form and assume the fixed point set X of the action has dimension $\frac{1}{2}\dim M$. Then there exists a neighbourhood of X in M which is isomorphic* (*as a hyperkähler manifold*) *to a neighbourhood of the zero section of T^*X endowed with the hyperkähler structure constructed from the induced Kähler metric on X.*

We then deduce

Corollary 1. *Any hyperkähler metric on a neighbourhood of the zero section of the cotangent bundle of a real-analytic Kähler manifold which is compatible with the canonical holomorphic symplectic structure, extends the given Kähler metric and for which the S^1-action by scalar multiplication in the fibres is isometric is unique in a* (*possibly smaller*) *neighbourhood of the zero section.*

The uniqueness result has been obtained independently by Kaledin [7] using different methods.

The proof relies on the twistor correspondence for hyperkähler manifolds. We shall identify the two sets of twistor data by studying certain features of the twistor spaces involved. This can only be made more precise once we recalled some statements about the twistor correspondence for hyperkähler manifolds in general and some facts about the twistor space construction in [3] in particular. The main ingredients will be Lemmas 4 and 5 which are stated in the next section.

2. Background material

2.1. The twistor correspondence

Recall that the twistor space Z of a hyperkähler manifold M as a differentiable manifold is just the product $M \times \mathbb{C}P^1$. The complex structure **I** on Z is given by (I_λ, J_0) on $T_{(m,\lambda)}Z \cong T_m M \oplus T_\lambda \mathbb{C}P^1$ where J_0 is the standard complex structure on $\mathbb{C}P^1$ and $I_\lambda = aI + bJ + cK$ where $\lambda \in \mathbb{C}P^1$ corresponds to $(a, b, c) \in S^2 \subset \mathbb{R}^3$ using stereographic projection, i.e., $\lambda = \frac{b-ic}{1-a}$.

The twistor space Z has a holomorphic twistor projection $p : Z \to \mathbb{C}P^1$ by projecting onto the second factor. Sections of p are called twistor lines.

The twistor space Z carries a real structure τ (i.e., an antiholomorphic involution), and each point in $m \in M$ gives rise to a real twistor line, i.e., a section $s_m : \mathbb{C}P^1 \to Z$ such that $s_m(-\bar{\lambda}^{-1}) = \tau(s_m(\lambda))$.

Hyperkähler manifolds are holomorphic symplectic in many different ways, for example if ω_1, ω_2 and ω_3 denote the Kähler forms with respect to I, J and K respectively then $\omega_2 + i\omega_3$ is a holomorphic symplectic form with respect to I. More generally, the hyperkähler metric gives rise to a 2-form on Z twisted by a section of $\mathcal{O}(2)$, the restriction of which to each fibre $p^{-1}(\lambda)$ is a holomorphic symplectic form.

It is well known that the twistor data always determines a (possibly indefinite) hyperkähler manifold. More precisely,

Theorem B (see Hitchin, Karlhede, Lindström and Roček [6]). *Let Z^{2n+1} be a complex manifold such that*

(i) *Z is a holomorphic fibre bundle $p : Z \to \mathbb{C}P^1$ over the projective line,*
(ii) *the bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$,*
(iii) *there exists a holomorphic section ω of $\Lambda^2 T_F^*(2) = \Lambda^2 T_F^* \otimes p^*\mathcal{O}(2)$ (where $T_F = \ker(dp : TZ \to T\mathbb{C}P^1)$ is the tangent bundle along the fibre) defining a symplectic form on each fibre,*
(iv) *Z has a real structure τ compatible with* (i), (ii) *and* (iii) *and inducing the antipodal map on $\mathbb{C}P^1$.*

Then the parameter space of real sections is a $4n$-dimensional manifold with a natural hyperkähler metric for which Z is the twistor space.

In condition (iv) "compatible" means that the metric constructed from the twistor data is positive definite or, more generally, has the desired signature.

2.2. The twistor space for a hyperkähler metric on the cotangent bundle

Next we have to recall some of the details of the twistor space construction in [3].

Let X be a real analytic Kähler manifold, then there exists (see [3, Lemma 1]) a holomorphic coordinate system such that $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ are real analytic coordinates with the Kähler form ω given by $\omega = \sum_{i,j} h_{ij}\, dz_i \wedge d\bar{z}_j$ with real-analytic coefficient functions h_{ij}.

Locally, the complexification X^c of the Kähler manifold X is given by holomorphic coordinates $z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n$, and the complexified Kähler form is

$$\omega_c = \sum_{i,j=1}^{n} h_{ij}\, dz_i \wedge d\tilde{z}_j \tag{1}$$

where we think of h_{ij} as an analytic continuation of the above coefficients h_{ij} of ω for $\tilde{z}$ near $\bar{z}$. Then ω_c is a holomorphic symplectic form on X^c.

On X^c there is a real structure $\sigma : X^c \to X^c$ given locally by

$$\sigma(z_1, \dots, z_n, \tilde{z}_1, \dots, \tilde{z}_n) = (\bar{\tilde{z}}_1, \dots, \bar{\tilde{z}}_n, \bar{z}_1, \dots, \bar{z}_n).$$

The complexification X^c carries two natural foliations given locally by $z_i = \text{const}$ for $1 \leqslant i \leqslant n$ and $\tilde{z}_i = \text{const}$ respectively. We call these the $+$ foliation (respectively – foliation). On a small open subset their leaves form submanifolds of X^c which are obviously Lagrangian with respect to ω_c given in (1). Each leaf Λ carries a natural flat torsion-free affine connection ∇, cf. [8].

For each foliation we obtain a vector bundle V over the space of leaves B by defining the fibre over $b \in B$ as $V_b = \{f : \Lambda_b \to \mathbb{C};\ \nabla(df) = 0\}$ where Λ_b is a simply connected neighbourhood in the leaf determined by b. The dual bundles V_+^* (for the $+$ foliation) and V_-^* (for the $-$ foliation) form the two halves of the twistor space. On each half, the twistor projection is given by evaluation at the constant affine linear function 1 and for each $x \in X^c$ a twistor line is given by the map $\phi : X^c \times \mathbb{C} \to V^*$, $\phi(x, \lambda) = \lambda\delta_x$ where δ_x is the evaluation map at x of an affine linear function.

Each half is locally standard (Lemma 2 in [3]), i.e. there exists a holomorphic coordinate system $q_1, \dots, q_n, p_1, \dots, p_n$ of X^c such that $1, p_1, \dots, p_n$ is a basis of affine linear functions and choosing coordinates $\alpha_0, \dots, \alpha_n$ with respect to the dual basis on V^* the twisted holomorphic symplectic form is given by $\sum_{j=1}^{n} dq_j \wedge d\alpha_j$ and the twistor projection is given by $p(q_1, \dots, q_n, \alpha_0, \dots, \alpha_n) = \alpha_0$.

The gluing of the two halves of the twistor space is provided by the maps ϕ_+ and ϕ_- which are holomorphic diffeomorphisms of a neighbourhood of X in X^c onto its image (Proposition 2 in [3]). For each $\lambda \in \mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$ the fibre $p^{-1}(\lambda)$ is biholomorphic to X^c and the twisted symplectic form satisfies $\lambda\omega_c = \phi_\lambda^*(\omega)$ where $\phi_\lambda : X^c \to p^{-1}(\lambda) \subset Z$ is given by $\phi_\lambda(x) = \phi(x, \lambda)$.

2.3. Hyperkähler manifolds with S^1-action

Now let M be a hyperkähler manifold with S^1-action as described in the introduction and fixed point set X, and let Z be the twistor space of M. The fixed point set X is a complex submanifold of M with respect to the complex structure I. Furthermore,

Lemma 1. (i) *$X \subset M$ is Lagrangian with respect to ω_c.*

(ii) *S^1 acts on the normal bundle $N_{X/M}$ of X in M by scalar multiplication.*

Proof. For $u = e^{i\theta} \in S^1$ and $v \in TM$ denote by $u.v$ the group action and by $e^{i\theta}v$ multiplication. Since X is the fixed point set of the action, $u.v = v$ for all $v \in TX$. Thus, for $v_1, v_2 \in TX$ and all $u \in S^1$

$$\omega_c(v_1, v_2) = \omega_c(u.v_1, u.v_2) = u.\omega_c(v_1, v_2) = e^{i\theta}\omega_c(v_1, v_2),$$

hence $\omega_c(v_1, v_2) = 0$, and (i) is shown.

For all $v \in TX$, $w \in N_{X/M}$ and $u = \mathrm{e}^{i\theta} \in S^1$

$$\omega_c(v, u.w) = \omega_c(u.v, u.w) = u.\omega_c(v, w) = \mathrm{e}^{i\theta}\omega_c(v, w) = \omega_c\big(v, \mathrm{e}^{i\theta}w\big),$$

thus, as X is Lagrangian, $u.w = \mathrm{e}^{i\theta}w$, and (ii) is proved. □

Taking the S^1-action on $\mathbb{C}P^1$ which fixes 0 and ∞ and using that the twistor space Z is diffeomorphic to $M \times S^2$ as a differentiable manifold, we can lift the action to Z. Its fixed point set contains two components, namely $X \subset p^{-1}(0)$ and $\overline{X} \subset p^{-1}(\infty)$ where $p: Z \to \mathbb{C}P^1$ is the twistor projection and $\overline{X}$ denotes X with the complex structure $-I$.

Lemma 2. *The S^1-action on Z is holomorphic.*

Proof. Denote by ξ the Killing field of the S^1-action, and recall that the complex structure $\mathbf{I}$ on Z is given by (I_λ, J_0) on $T_{(m,\lambda)}Z \cong T_mM \oplus T_\lambda\mathbb{C}P^1$ where J_0 is the standard complex structure on $\mathbb{C}P^1$. Obviously, $\mathcal{L}_\xi J_0 = 0$, and since S^1 acts by scaling on $\omega_2 + i\omega_3$ and is also isometric, $\mathcal{L}_\xi J = -K$ and $\mathcal{L}_\xi K = J$. Since the action on M is holomorphic with respect to I, $\mathcal{L}_\xi I = 0$.

Next we note that if $\lambda = \frac{b+ic}{1+a}$ is the standard affine parameter on $\mathbb{C}P^1 \setminus \{\infty\}$, $\xi(a) = 0$, $\xi(b) = -c$ and $\xi(c) = b$. Therefore,

$$\mathcal{L}_\xi I_\lambda = \mathcal{L}_\xi(aI + bJ + cK) = \xi(b)J + b\mathcal{L}_\xi J + \xi(c)K + c\mathcal{L}_\xi K = 0$$

hence the action on Z is holomorphic. □

Lemma 3. *The induced S^1-action on the normal bundle $N_{X/Z}$ of X in Z is given by scalar multiplication.*

Proof. Note that at a point $x \in X$

$$N_{X/Z_x} = N_{X/M_x} \oplus p^*T_{p(x)}\mathbb{C}P^1,$$

and the action in the additional normal direction has also weight 1. □

Remark 1. Since the vector field ξ generated by the S^1-action is holomorphic, $[\xi, I\xi] = I[\xi, \xi] = 0$, and we can define a local $\mathbb{C}^*$-action in a neighbourhood of $1 \in \mathbb{C}^*$ by $\mathrm{e}^{s+it}.p = \psi(s, \phi(t, p))$ where $\phi(t, \cdot)$ is the flow of ξ and $\psi(s, \cdot)$ is the flow of $I\xi$. This is well defined since the flows—which always exist near zero—commute. Since $\phi(t, \cdot)$ is the given S^1-action we can deduce that the above defines an action for all $\lambda \in \mathbb{C}^*$ with $|\lambda|$ near 1.

The following two lemmas which will be shown in the next two sections are the key to prove Theorem A.

Lemma 4. *There exists $c \in]0, 1[$ such that $Z|_{p^{-1}(\{\lambda \in \mathbb{C}; |\lambda| < c\})}$ and $Z|_{p^{-1}(\{\lambda^{-1} \in \mathbb{C}; |\lambda| < c\})}$ have a standard coordinate system, and can be identified with the flat model.*

By a standard coordinate system we mean in this context that there exist holomorphic coordinates $x_1, \ldots, x_n, w_0, \ldots, w_n$ of Z such that $p(x_1, \ldots, x_n, w_0, \ldots, w_n) = w_0$, the twisted holomorphic symplectic form is given by $\sum_{j=1}^n dx_j \wedge dw_j$ and S^1 acts trivially on x_j for all j and by multiplication by e^{it} on w_j for all j.

Lemma 5. *Each fibre over $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{0, \infty\}$ carries two transverse Lagrangian foliations and can be identified with a neighbourhood of X in its complexification X^c respecting the transverse foliations of the fibre and of X^c. Each point in X^c gives rise to a twistor line joining a point in $X \subset p^{-1}(0)$ and a point in $\overline{X} \subset p^{-1}(\infty)$.*

3. Standardisation of parts of the twistor space

In order to prove Lemma 4 we recall that the fixed point set X can be identified with a submanifold of the twistor space Z and use the flow lines of $\mathbf{I}\xi$ (where ξ is the Killing field of the S^1-action extended to Z and $\mathbf{I}$ is the complex structure of Z) to construct a holomorphic projection of a neighbourhood of X in Z onto X. Then we show that the inverse image of points under this projection gives rise to Lagrangian foliations in the fibres of the twistor space contained in this neighbourhood. The same process can be done for $\overline{X}$.

The main aim of this section is to prove

Proposition 1. *There exist holomorphic coordinates $w_0, \ldots, w_n, x_1, \ldots, x_n$ of Z near X such that $x_1, \ldots, x_n$ are S^1-invariant and coordinates for X for $w_0 = w_1 = \cdots = w_n = 0$, S^1 acts on $w_0, \ldots, w_n$ by multiplication with $\mathrm{e}^{i\theta}$ and the twistor projection is given by $p(w_0, \ldots, w_n, x_1, \ldots, x_n) = w_0$.*

Furthermore, the twisted symplectic form along the fibres is $\sum_{i=1}^n dx_i \wedge dw_i$ in these coordinates.

For the proof we need to find eigenfunctions of the Killing field ξ of the circle action on Z. We achieve this by first producing an S^1-invariant holomorphic submersion onto an open subset of $\mathbb{C}^n$ and then applying Poincaré's theorem (cf. [1]) to the vector field obtained from the Killing field, but now acting along the fibres of this map.

The Killing field ξ is a holomorphic vector field, i.e., $\mathcal{L}_\xi \mathbf{I} = 0$, hence it is the real part of a complex vector field with holomorphic coefficients. As we are only interested in holomorphic data we use this complex vector field instead of ξ itself in our calculations.

Lemma 6. *There exist holomorphic coordinates $(z_1, \ldots, z_{2n+1})$ of Z near X such that $z_1, \ldots, z_n$ are S^1-invariant and X corresponds to $z_{n+1} = \cdots = z_{2n+1} = 0$.*

Proof. We can find coordinates $v_1, \ldots, v_n, z_{n+1}, \ldots, z_{2n+1}$ of Z such that $dv_1, \ldots, dv_n$ are invariant under the induced action on the cotangent bundle at points in X which correspond to $z_{n+1} = \cdots = z_{2n+1} = 0$. Then we average over the compact group S^1 to obtain

$$z_j = \frac{1}{2\pi} \int\limits_{S^1} u.v_j \, du$$

for $1 \leqslant j \leqslant n$. At points in X, $dz_j = dv_j$, thus $z_1, \ldots, z_n$ locally define an S^1-invariant holomorphic submersion onto an open set of $\mathbb{C}^n$ giving the desired coordinate system. □

The Killing field now acts along the fibres of the projection map

$$(z_1, \ldots, z_{2n+1}) \mapsto (z_1, \ldots, z_n).$$

As the induced action on the normal bundle is given by scalar multiplication (see Lemma 3) we can assume without loss of generality that the Killing field along the fibres is given by

$$\xi_F = i \sum_{j=n+1}^{2n+1} z_j \frac{\partial}{\partial z_j} + \text{higher order terms.}$$

Next we establish the main lemma.

Lemma 7. *If ξ denotes the Killing field of the S^1-action on Z, then there exists a coordinate system $(z_1, \ldots, z_n, y_1, \ldots, y_{n+1})$ of Z near X such that $\xi(z_i) = 0$ for all $1 \leqslant i \leqslant n$ and $\xi(y_j) = i y_j$ for all $1 \leqslant j \leqslant n+1$.*

Proof. Choosing a coordinate system as in Lemma 6, $\xi(z_i) = 0$ for $1 \leqslant i \leqslant n$.

Now we consider the vector field ξ_F for any fibre determined by $(z_1, \ldots, z_n)$. The linear part of ξ_F at the singular point $z_{n+1} = \cdots = z_{2n+1} = 0$ has (multiple) eigenvalue $+i$. We now apply Poincaré's theorem in the following form (cf. [1, p. 190]) which also holds in the case of multiple eigenvalues.

Theorem C (Poincaré). *If the eigenvalues of the linear part of a holomorphic vector field at a singular point belong to the Poincaré domain* (*i.e., the convex hull of the eigenvalues does not contain zero*) *and are nonresonant, then the field is biholomorphically equivalent to its linear part in a neighbourhood of the singular point.*

Therefore we can find coordinate functions $y_1, \ldots, y_{n+1}$ (which depend holomorphically on $z_1, \ldots, z_{2n+1}$) such that

$$\xi_F = i \sum_{j=1}^{n+1} y_j \frac{\partial}{\partial y_j},$$

hence $\xi(y_j) = i y_j$ for $1 \leqslant j \leqslant n+1$, and $z_1, \ldots, z_n, y_1, \ldots, y_{n+1}$ is the desired coordinate system. □

We are now ready to show Proposition 1.

Proof. We start with a local coordinate system $z_1, \ldots, z_n, y_1, \ldots, y_{n+1}$ as in Lemma 7.

We note that all other coordinate systems $\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{y}_1, \ldots, \tilde{y}_{n+1}$ such that $\tilde{z}_1, \ldots, \tilde{z}_n$ are S^1-invariant coordinates and the action is linear on $\tilde{y}_1, \ldots, \tilde{y}_{n+1}$, can be obtained from $z_1, \ldots, z_n, y_1, \ldots, y_{n+1}$ by a transformation of the form

$$\tilde{z}_i = f_i(z_1, \ldots, z_n),$$
$$\tilde{y}_j = \sum_{l=1}^{n+1} g_{jl}(z_1, \ldots, z_n) y_l$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n+1$.

As the twistor projection p is equivariant and S^1 acts by multiplication on the standard affine parameter on $\mathbb{C}P^1 \setminus \{\infty\}$, p is of the form

$$p(z_1, \ldots, z_n, y_1, \ldots, y_{n+1}) = \sum_{i=1}^{n+1} h_i(z_1, \ldots, z_n) y_i,$$

hence by defining $\tilde{y}_{n+1} = p(z_1, \dots, z_n, y_1, \dots, y_{n+1})$ and $\tilde{y}_j = y_j$ and setting $x_i = z_i$ for $1 \leqslant i \leqslant n$ we have found coordinates $x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_{n+1}$ such that the twistor projection is given by $\tilde{y}_{n+1}$ and $x_1, \dots, x_n, \tilde{y}_1, \dots, \tilde{y}_n$ are coordinates in the fibres.

Locally in each fibre, the twisted symplectic form can be written as $\omega = -d\theta$ with a 1-form θ along the fibres which depends on the parameter $\tilde{y}_{n+1}$. As S^1 acts by scaling on ω it must also act by scaling on θ, hence generally θ will be of the form

$$\theta = \sum_{j=1}^{n+1} \tilde{y}_j \alpha_j + \sum_{k=1}^{n} f_k \, d\tilde{y}_k,$$

where $\alpha_j = \sum_{i=1}^n a_{ij}(x_1, \dots, x_n)\, dx_i$ and f_k are functions of $x_1, \dots, x_n$ with parameter $\tilde{y}_{n+1}$. Since we can replace θ by $\theta - dF$ for a function F without changing ω, choosing $F = \sum_{k=1}^n f_k \tilde{y}_k$, we can assume without loss of generality that θ is of the form

$$\theta = \sum_{j=1}^{n+1} \tilde{y}_j \alpha_j = \sum_{j=1}^{n+1} \sum_{i=1}^{n} \tilde{y}_j a_{ij}\, dx_i.$$

We set $w_0 = \tilde{y}_{n+1}$ and $w_i = \sum_{j=1}^{n+1} a_{ij} \tilde{y}_j$, then the map $(\tilde{y}_1, \dots, \tilde{y}_{n+1}) \mapsto (w_0, \dots, w_n)$ is locally biholomorphic because its derivative is given by the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ a_{11} & \cdots & a_{1n} & a_{01} \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{0n} \end{pmatrix}$$

which is non-degenerate as the matrix $A = (a_{ij})_{i,j=1}^n$ is non-degenerate since ω is symplectic.

Hence $x_1, \dots, x_n, w_0, \dots, w_n$ are holomorphic coordinates such that $w_0, \dots, w_n$ are homogeneous of degree one, the twistor projection is given by w_0 and the twisted holomorphic symplectic form is given by $\omega = -d\theta = -d(\sum_{i=1}^n w_i dx_i) = \sum_{i=1}^n dx_i \wedge dw_i$. $\square$

The holomorphic projection is now given by the limit points of flow lines of $\mathbf{I}\xi$. The flow lines can be thought of as parts of $\mathbb{R}^+$-orbits and the limit points correspond to $0 \in \mathbb{R}$. For more on this point of view see Remark 1.

To make this more precise we take a coordinate system as in Proposition 1. In these coordinates $\mathbf{I}\xi$ is given by

$$\mathbf{I}\xi = -\sum_{j=0}^{n} w_j \frac{\partial}{\partial w_j},$$

hence the integral curves $\phi_t(\underline{x}, \underline{w})$ with $\phi_0(\underline{x}, \underline{w}) = (\underline{x}, \underline{w})$ are

$$\phi_t(\underline{x}, \underline{w}) = \left(\underline{x}, \mathrm{e}^{-t} \underline{w}\right),$$

and we define the projection $q : U \to X$ by

$$q(\underline{x}, \underline{w}) = \lim_{t \to \infty} \phi_t(\underline{x}, \underline{w}) = (\underline{x}, 0) \in X.$$

Obviously, for all $\underline{x} \in X$ and all $\lambda \in \mathbb{C}P^1$ such that $p^{-1}(\lambda) \cap U \neq \emptyset$, $q^{-1}(\underline{x}) \cap p^{-1}(\lambda)$ is a submanifold of $p^{-1}(\lambda)$ which is Lagrangian with respect to the symplectic form along the fibres given in coordinates by $\sum_{i=1}^{n} dx_i \wedge dw_i$.

As $p(U) \supset \{\lambda \in \mathbb{C}; |\lambda| < c\}$ for some $0 < c < 1$ (by the open mapping theorem) we can identify $Z|_{p^{-1}(\{\lambda \in \mathbb{C}; |\lambda| < c\})}$ with the standard flat model using these special coordinates.

This identification respects the foliations in the fibres and the symplectic structure along the fibres.

4. The identification of the fibres over $\mathbb{C}^*$

This section deals with the proof of Lemma 5. We first have to identify the complexification X^c of X in terms of twistor lines.

Lemma 8. *The complexified manifold X^c can be identified with the set of all twistor lines $s : \mathbb{C}P^1 \to Z$ such that $s(0) \in X$ and $s(\infty) \in \overline{X}$.*

Proof. From the theory of the twistor correspondence we know that the complexified hyperkähler manifold M^c corresponds to the set of all twistor lines.

As $X \subset M$ is the fixed point set of the S^1-action, the complexified manifold $X^c \subset M^c$ is the fixed point set of the complexified action and therefore consists of all twistor lines which are $\mathbb{C}^*$-invariant (under the local $\mathbb{C}^*$-action introduced in Remark 1).

Hence $s(0)$ and $s(\infty)$ must be fixed points, i.e., $s(0) \in X$ and $s(\infty) \in \overline{X}$, see Section 2.3. Twistor lines with this property are $\mathbb{C}^*$-invariant because otherwise the action would give rise to a 1-parameter deformation of the original twistor line keeping $s(0)$ and $s(\infty)$ fixed and thus to a section of the normal bundle N vanishing at 0 and ∞; but as $N = \mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$ and any non-zero section of $\mathcal{O}(1)$ vanishes only at one point, this section must be identically zero. □

Below we will use the following straightforward result from algebraic geometry which will be proved in Appendix A.

Let Z be a complex manifold of dimension $2n + 1$ and Y_1 and Y_2 complex submanifolds of dimension r and s respectively with $r + s \leqslant 2n$ and $Y_1 \cap Y_2 = \emptyset$.

Let C be a rational curve in Z intersecting Y_i transversally in exactly one point y_i, $i = 1, 2$. We assume that the normal bundle N of C in Z is $\mathbb{C}^{2n} \otimes p^*\mathcal{O}(1)$.

We blow up Y_1 and Y_2 and consider the proper transform $\widehat{C}$ of C in the blow-up $\widehat{Z}$.

Proposition 2. *The normal bundle $\widehat{N}$ of $\widehat{C}$ in $\widehat{Z}$ is*

$$\widehat{N} = \mathbb{C}^{r+s} \oplus \mathbb{C}^{2n-(r+s)} \otimes p^*\mathcal{O}(-1).$$

Now we are ready to establish the identification which proves Lemma 5.

Proposition 3. *The map*

$$\begin{aligned} F : X^c \times \mathbb{C}^* &\to Z \\ (x, \lambda) &\mapsto s_x(\lambda) \end{aligned}$$

where s_x is the twistor line corresponding to x defines an isomorphism from $U \times \mathbb{C}^*$ onto its image for a neighbourhood U of X in X^c.

Before we prove Proposition 3 we establish the following

Lemma 9. *For each $\lambda \in \mathbb{C}^*$ and any $x \in X$ the derivative of the map*

$$\begin{aligned} F_\lambda : X^c &\to p^{-1}(\lambda) \\ x &\mapsto F(x,\lambda) \end{aligned}$$

at x is an isomorphism.

Proof. Since the tangent space at a point in M^c can be identified with the sections of the normal bundle of the twistor line, we can identify $T_x X^c$ with those sections s of the normal bundle which satisfy $s(0) \in T_{s_x(0)} X$ and $s(\infty) \in T_{s_x(\infty)} \overline{X}$.

A point $x \in X \subset X^c$ corresponds to the real twistor line s_x with $s_x(0) = x$ and $s_x(\infty) = \bar{x}$.

We consider the blow-up $\widehat{Z}$ of Z along X and $\overline{X}$ and let $\widetilde{X}^c$ be the set of all sections $s : \mathbb{C}P^1 \to \widehat{Z}$ obtained by deforming the proper transform of $s_x(\mathbb{C}P^1)$, $x \in X^c$. For these sections $s(0) \in E_X$ and $s(\infty) \in E_{\overline{X}}$ where E_X and $E_{\overline{X}}$ are the exceptional divisors. Define

$$\widetilde{F}_\lambda : \widetilde{X}^c \to p^{-1}(\lambda), s \mapsto s(\lambda)$$

then for $\lambda \in \mathbb{C}^*$, $\widetilde{F}_\lambda$ is an isomorphism if and only if F_λ is because the blowing-down map is an isomorphism outside the exceptional divisors.

But using Proposition 2 we conclude that the normal bundle $\widehat{N}$ in the blow-up is trivial: $\widehat{N} = \mathbb{C}^{2n}$, hence we can prescribe any given vector in $T_{s(\lambda)} p^{-1}(\lambda)$ at λ, and $F'_\lambda(x)$ is an isomorphism. □

Proof of Proposition 3. Since $\frac{\partial F}{\partial \lambda}(x,\lambda) \in T_{s_x(\lambda)}(s_x(\mathbb{C}P^1))$ is non-zero and

$$\begin{aligned} T_{s_x(\lambda)} Z &= N_{s_x(\lambda)} \oplus T_{s_x(\lambda)}\big(s_x\big(\mathbb{C}P^1\big)\big) \\ &= T_{F_{s_x(\lambda)}} \oplus T_{s_x(\lambda)}\big(s_x\big(\mathbb{C}P^1\big)\big), \end{aligned}$$

using Lemma 9 and the inverse function theorem we deduce that F is a local isomorphism of $U \times \mathbb{C}^*$ where U is a neighbourhood of X in X^c.

Furthermore, F is injective because, for any $x \in X^c$, $s_x(\mathbb{C}^*)$ is a $\mathbb{C}^*$-orbit, and this completes the proof. □

We now describe a procedure by which the standard foliations of X^c give rise to Lagrangian foliations in the fibres $p^{-1}(\lambda)$, $\lambda \in \mathbb{C}^*$, of Z.

Let s_x be the real twistor line with $s(0) = z \in X$ and $s(\infty) = \bar{z} \in \overline{X}$. We consider deformations s of s_x such that $s(0) = z$ is fixed and $s(\infty) \in \overline{X}$. These lines correspond to the foliation in X^c locally given by $z_i = \text{const}$ if we assume that the complexification X^c has coordinates $(z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n)$.

Similarly, the foliation $\tilde{z}_i = \text{const}$ corresponds to twistor lines with $s(0) \in X$ and $s(\infty) = \tilde{z} \in \overline{X}$ fixed.

Proposition 4. *The foliations of X^c give rise to Lagrangian foliations in each fibre over $\mathbb{C}^*$ of the twistor space.*

Proof. Without loss of generality we consider the case $z_i = \text{const}$, and the real twistor line C joining z and $\bar{z}$.

We blow-up Z along z and $\overline{X}$ and look at the proper transform $\widehat{C}$ of C in the blow-up $\widehat{Z}$. By Proposition 2 the normal bundle $\widehat{N}$ of $\widehat{C}$ in $\widehat{Z}$ will be

$$\widehat{N} = \mathbb{C}^n \oplus \mathbb{C}^n \otimes p^*\mathcal{O}(-1).$$

Therefore, $H^1(\widehat{N}) = 0$ and $H^0(\widehat{N}) \cong \mathbb{C}^n$, hence the deformations of this line sweep out n-dimensional subspaces in each $p^{-1}(\lambda)$, $\lambda \in \mathbb{C}^*$.

If ω denotes the symplectic form along $p^{-1}(0)$ then the symplectic form H along $p^{-1}(\lambda)$ evaluated at two tangent vectors $v = a\lambda + b$ and $w = c\lambda + d$ (here we use the identification of tangent vectors with sections of the normal bundle) is given by

$$H(v, w) = \lambda^2\omega(a, c) + \lambda\big(\omega(a, d) + \omega(b, c)\big) + \omega(c, d).$$

Tangent vectors to a leaf of the foliation correspond to sections s of the normal bundle vanishing at $\lambda = 0$, i.e., $b = d = 0$ and $s(\infty) \in \overline{X}$, i.e., $a, c \in TX$, and hence $\omega(a, c) = 0$ as $X \subset p^{-1}(0)$ is Lagrangian.

Therefore $H(v, w) = 0$ if v and w are tangent to a leaf, and thus the leaves are isotropic. □

5. Proof of Theorem A

In [3] we constructed the twistor space Z' for a hyperkähler manifold M' in a neighbourhood of the zero section of the cotangent bundle of any real-analytic Kähler manifold X, see Section 2.2.

Using the coordinate system of Proposition 1 we note that, under the assumptions of Theorem A, a neighbourhood of X in $p^{-1}(0)$ (which corresponds to M together with the complex structure I) can be identified (as holomorphic symplectic manifold with Lagrangian foliation) with a neighbourhood of the zero section of the cotangent bundle T^*X.

Lemmas 4 and 5 now imply the equivalence of the twistor spaces Z and Z' and their twistor lines, hence the hypercomplex structures of M and M' are the same. The holomorphic symplectic structures are equivalent (up to a constant), and thus the metrics are equivalent up to a constant. But since both metrics restrict to the same Kähler metric on X the metrics are actually the same, which concludes the proof of Theorem A and Corollary 1 is obvious.

Theorem A and its corollary allow us to deduce information about the (in)completeness of some of the hyperkähler metrics on cotangent bundles obtained in [3] or [7].

For example, Hitchin [5] has constructed a complete hyperkähler metric on the moduli space of Higgs bundles over a Riemann surface. This moduli space contains the moduli space of stable bundles as a dense open subset, and the hyperkähler metric restricts to the natural Kähler metric on the moduli space of stable bundles on the zero section. As the S^1-action satisfies the assumptions of Theorem A we deduce that the hyperkähler extension on the cotangent bundle induced by the natural metric on the moduli space of stable bundles is incomplete.

Finally we look at an example, namely how different S^1-invariant Kähler metrics on the projective line give rise to hyperkähler metrics with quite different completeness properties.

Example 1. We use the description of cyclic ALE metrics as a hyperkähler quotient of $\mathbb{H}^m \times \mathbb{H}$ by T^m given in [4].

If $(q_1, \dots, q_m, q)$ are coordinates in $\mathbb{H}^m \times \mathbb{H}$ and $(t_1, \dots, t_m) \in T^m$, the action is given by

$$(q_1, \dots, q_m, q) \mapsto \left(q_1 \mathrm{e}^{it_1}, \dots, q_m \mathrm{e}^{it_m}, q\mathrm{e}^{i\sum_{a=1}^m t_a}\right)$$

and the level sets of the moment map are given by $\mathbf{r}_a + \mathbf{r} = \mathbf{x}_a \in \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ where $\mathbf{r}_a = q_a i \bar{q}_a$ and $\mathbf{r} = q i \bar{q}$.

There is an additional triholomorphic S^1-action on $\mathbb{H}^m \times \mathbb{H}$ given by $(q_1, \dots, q_m, q) \mapsto (q_1, \dots, q_m, q\mathrm{e}^{it})$ which commutes with the T^m-action and preserves the level set and thus descends to triholomorphic action on the quotient.

If we write $q_a = z_a + j\bar{w}_a$ where $z_a, w_a \in \mathbb{C}$ and similarly $q = z + j\bar{w}$ then we can write down another isometric S^1-action on $\mathbb{H}^m \times \mathbb{H}$ given by

$$e(z_1 + j\bar{w}_1, \dots, z_m + j\bar{w}_m, z + j\bar{w}) \mapsto \left(z_1 + j\mathrm{e}^{it}\bar{w}_1, \dots, z_m + j\mathrm{e}^{it}\bar{w}_m, z + j\mathrm{e}^{it}\bar{w}\right)$$

which also commutes with the T^m-action. This action is holomorphic with respect to I and scales the corresponding holomorphic symplectic form on $\mathbb{H} \times \mathbb{H} \cong T^*\mathbb{C}^{m+1}$. This action descends to an isometric and I-holomorphic action on the quotient which scales the holomorphic symplectic form with respect to I if and only if the level set is preserved, i.e., if and only if the centres satisfy $\mathbf{x}_a = (x_a, 0, 0)$. Its fixed point set is the quotient by T^m of all orbits in $\mathbb{H}^m \times \mathbb{H}$ which are preserved by the action; this set contains, for example, the 3-spheres given by $|z_a|^2 + |z|^2 = x_a$, hence the fixed point set in the ALE spaces contains projective lines $\mathbb{C}P^1$ which possess an induced S^1-invariant Kähler metric.

Taking one of these projective lines and applying the constructions of [3] or [7] yields a hyperkähler metric in a neighbourhood of the zero section of $T^*\mathbb{C}P^1$ which must coincide by Theorem A with the restriction of the ALE metric. As the ALE space is complete (as a hyperkähler quotient of a complete manifold by a compact group), but its topology for $m > 1$ different from the topology of $T^*\mathbb{C}P^1$, the induced hyperkähler metric on a neighbourhood in $T^*\mathbb{C}P^1$ must be incomplete and the ALE space is a possible completion.

Furthermore, the Fubini–Study metric on $\mathbb{C}P^1$ gives rise to the Eguchi-Hanson metric on the cotangent bundle (see [2]); it is complete and corresponds to the case $m = 1$ in the above description. Hence using Theorem A, the S^1-invariant Kähler metrics on $\mathbb{C}P^1$ induced from the ALE metric for $m > 1$ must be different from the Fubini–Study metric.

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Appendix A

Now we shall prove Proposition 2. We use the following lemma which will be shown below.

Lemma 10. *There is an exact sequence of sheaves*

$$0 \to \widehat{N} \to N \xrightarrow{q} N_{C\cup Y_1/Z,y_1} \oplus N_{C\cup Y_2/Z,y_2} \to 0$$

where the map $\widehat{N} \to N$ is induced by the blowing-down map, $N_{C\cup Y_i/Z,y_i}$ is the skyscraper sheaf supported at y_i spanned by the directions normal to C and Y_i, $i = 1, 2$, and the map q is the evaluation at y_1 and y_2 respectively followed by projection onto the directions normal to C and Y_1, respectively C and Y_2.

As $N = \mathbb{C}^{2n}(1)$ the above sequence gives rise to a long exact sequence in cohomology

$$0 \to H^0(\widehat{N}) \to \mathbb{C}^{4n} \xrightarrow{q} \mathbb{C}^{2n-r} \oplus \mathbb{C}^{2n-s} \to H^1(\widehat{N}) \to 0$$

q is onto because any section of N is of the form $a\lambda + b$ with $a, b \in \mathbb{C}^{2n}$, thus we can prescribe any value in any direction at any two given points of the curve.

Therefore $H^1(\widehat{N}) = 0$ and $\dim H^0(\widehat{N}) = r + s$, and by the Riemann–Roch formula $\dim H^0(\widehat{N}) - \dim H^1(\widehat{N}) = \deg \widehat{N} + \operatorname{rank} \widehat{N} \cdot (1 - g)$, i.e., $\deg \widehat{N} = r + s - 2n$. By the Birkhoff–Grothendieck theorem,

$$\widehat{N} = \bigoplus_{i=1}^{2n} \mathcal{O}(m_i), \quad m_i \in \mathbb{Z}, \ \sum_{i=1}^{2n} m_i = r + s - 2n.$$

As $H^1(\widehat{N}) = 0$, $m_i \geqslant -1$ for all $1 \leqslant i \leqslant 2n$.

Claim. $m_i \leqslant 0$ *for all* $1 \leqslant i \leqslant 2n$.

Suppose $m_i =: m \geqslant 2$ for some i. Then the map $\widehat{N} \to N$ induces a homomorphism $\mathcal{O}(m) \to \mathbb{C}^{2n}(1)$ which must be zero as $H^0(\mathbb{C}^{2n}(1-m)) = 0$ for $m \geqslant 2$. Therefore $\mathcal{O}(m)$ would lie in the kernel of $\widehat{N} \to N$ contradicting injectivity.

Suppose $m_i = 1$ for some i. Any map $\mathcal{O}(1) \to \mathbb{C}^{2n}(1)$ is, on the level of sections, of the form $s \mapsto s \otimes v$ for some fixed $v \in \mathbb{C}^{2n}$. $s \otimes v$ must vanish at two points for all s, hence $v = 0$ and $\mathcal{O}(1)$ lies in the kernel of $\widehat{N} \to N$ giving a contradiction, and the claim is established.

Since $m_i \in \{0, -1\}$ and $\sum_{i=1}^{2n} m_i = r + s - 2n$,

$$\widehat{N} = \mathbb{C}^{r+s} \oplus \mathbb{C}^{2n-r+s} \otimes \mathcal{O}(-1).$$

To conclude the proof we have to prove Lemma 10.

Proof. We consider the local situation near $y := y_1$. A similar argument can be done near y_2. Outside y_1 and y_2 the blowing-down map p gives rise to an isomorphism $\widehat{N} \to N$.

Let $z_1, \ldots, z_{2n+1}$ be coordinates for Z near y such that $Y := Y_1$ is given by $z_{r+1} = \cdots = z_{2n+1} = 0$, and C is given by $z_1 = \cdots = z_{2n} = 0$, hence y corresponds to $\underline{z} = 0$. On the blow-up $\widehat{Z}$ of Z along Y we construct a coordinate system

$$(z_1, \ldots, z_r, \mu_1, \ldots, \mu_{2n-r}, z_{2n+1})$$

near the intersection of $\widehat{C}$ with the exceptional divisor where $(\mu_1, \ldots, \mu_{2n-r})$ are affine coordinates for $\mathbb{C}P^{2n-r}$ and the blowing-down map is given by

$$p(z_1, \ldots, z_r, \mu_1, \ldots, \mu_{2n-r}, z_{2n+1}) = (z_1, \ldots, z_r, z_{2n+1}\mu_1, \ldots, z_{2n+1}\mu_{2n-r}, z_{2n+1}).$$

Therefore

$$p_*\frac{\partial}{\partial z_i}=\frac{\partial}{\partial z_i}\quad\text{for } 1\leqslant i\leqslant r,$$
$$p_*\frac{\partial}{\partial \mu_j}=z_{2n+1}\frac{\partial}{\partial z_{r+j}}\quad\text{for } 1\leqslant j\leqslant 2n-r$$

and

$$p_*\frac{\partial}{\partial z_{2n+1}}=\frac{\partial}{\partial z_{2n+1}}+\sum_{l=0}^{2n-r}\mu_l\frac{\partial}{\partial z_{r+l}}.$$

p_* is an isomorphism except when $z_{2n+1}=0$, thus p_* is an isomorphism of the normal bundles except at y.

Therefore, $\widehat{N}\to N$ is injective as map of sheaves, and the quotient is a sheaf supported at y and spanned by $\frac{\partial}{\partial z_{r+1}},\ldots,\frac{\partial}{\partial z_{2n}}$, i.e., the directions normal to C and Y and the map q is as claimed. □

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